

A brief review on uncertainty principle and its variants

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Abstract

In this paper, using two different formulations, we attempted to first provide an overview of mathematical interpretations of Heisenberg's uncertainty principle. This was done in order to emphasise the mathematical justification for the uncertainty principle's existence. First, we went over in detail the matrix formulation found in Schrodinger's original paper on uncertainty. Next, we reviewed the mathematical proof of the uncertainty principle in a Fourier series formulation and examined the results. We attempted to provide a comprehensive overview of the modified forms of the uncertainty relation in the Quantum Gravitational, Relativistic, and Entropic domains along with other variants after realizing that the most well-known uncertainty relation is not universal. All of these concepts also demonstrate a great deal of potential for use in the emerging new technologies. Our goal was to gather all of these dispersed variations of the uncertainty principle in one location. The current state of these theories, as well as their applications and limitations, are thoroughly discussed.

Keywords: Quantum optics, Generalised uncertainty relations, Relativistic uncertainty relation, Entropic uncertainty relation

1. Introduction

The Heisenberg's Uncertainty Principle is an inevitable consequence of the existence of quantum physics. The theory formulated by Werner Heisenberg in 1927 gave a new insight to the world of Physics. The principle set a limit on the extent of precision that one can attain in a quantum measurement.

It is known that in Quantum Physics, an observable can be represented by an operator. When such an operator acts on a system represented by a wavefunction or state, the eigenvalue of the operator in that state represents the physical observable. As the observables are real numbers, the corresponding operator must be Hermitian. If the system is not an eigenstate of the operator, we get uncertainties in the measurement of the observable. According to the Heisenberg Uncertainty principle the product of uncertainties in two non-commutable Hermitian operators is related by an inequality [15]. Suppose we have two Hermitian operators \hat{A} , \hat{B} and the uncertainties are represented as ΔA and ΔB . According to the principle, if $[A, B] \neq 0$, then

$$(\Delta A)^2 (\Delta B)^2 \geq \left(\frac{1}{2i} \langle [A, B] \rangle \right)^2 \quad (1)$$

It is noteworthy that the left side of the inequality is positive and real valued. To satisfy the inequality the right side of the inequality must also be real and non-negative. The expressions for the fluctuations

$$(\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2 \quad (2)$$

$$(\Delta B)^2 = \langle B^2 \rangle - \langle B \rangle^2 \quad (3)$$

It should also be noted that to satisfy the above equations, $\langle A^2 \rangle$ must be greater than $\langle A \rangle^2$ which is also true for $\langle B^2 \rangle$ and $\langle B \rangle^2$.

The theory is strongly supported by mathematics [1, 2, 9, 5]. Different mathematical formulations confirm the existence and validity of the principle. This motivated us to have a review of some of the different mathematical interpretations that leads to the confirmation of the principle. The mathematical derivation of the principle using matrices as operators can be found in [1]. Extensions and modifications of the principle can be found in later times in a more general form [2, 9, 10, 16-31, 34-39]. It is already proven that the inequality reduces to its lower bound for certain special states (Minimum Uncertainty States) which give rise to various interesting quantum optical phenomena like coherent states, squeezed states etc [3, 4, 6-8].

In this report, we have gone through Schrodinger's original matrix formulation [2] of the uncertainty principle using techniques like the splitting of operator products and the Schwartz inequality. The use of Fourier series to prove the uncertainty relation [5] will be thoroughly investigated. We have chosen an example to examine the validity of the principle on a coherent state which is a minimum uncertainty state by measuring the fluctuations in position and momentum. The coherent state will be taken as a special interference of number states.

We have presented a detailed review of the variants of the well-known form of the uncertainty principle on the ground of its validity at different conditions like Planck's length scale [16-25], relativistic domain [26-31] etc. Entropic form of the uncertainty principle has also been discussed which plays an important role in recent quantum communication technologies. Variance based Uncertainty relation, sum of square of uncertainties based relation, multiple observable based uncertainty relations are also mentioned. The need for unification of these different domains is highlighted in the later sections.

The Uncertainty principle and its variants is a huge topic in itself. There can be found numerous literatures, books, notes etc. which review the principle taking into account only one variation or modification of the principle. We therefore aim to give a comprehensive treatment of all of these topics in one place, with the hope of benefiting some of the quickly emerging technologies that exploit quantum information. Also, it helps to have a comparative study of the different cases which eventually leads to the questions like a unified form, a generalized form of the uncertainty principle, its invariance etc. The paper is arranged as follows: Section 2 contains the mathematical treatments of the Uncertainty principle. In sub-section 2.1 we have discussed the mathematical proof of uncertainty principle using matrix formulation. In sub-section 2.2 we have discussed the same principle using Fourier analysis. In sub-section 2.3, we have tested the uncertainty principle in a coherent state as an example. In section 3 we reviewed the different variations of the uncertainty principle. To be specific, in sub-section 3.1 we have discussed the Generalized Uncertainty Principle. Subsection 3.2 contains the relativistic scenario of the uncertainty principle, 3.3 contains the entropic form. In section 4 we have presented a brief note on the experimental progress related to the measurement of uncertainty principle just to get an idea of the non-theoretical realm.

2. Mathematical interpretation of uncertainty principle

The theory of the uncertainty principle is strongly backed by mathematical formulations. In literature there exist hundreds of different formulations which confirm the existence of the uncertainty principle. In this paper, we shall concentrate only on two of them: the matrix formulation and the Fourier analysis.

2.1 Uncertainty principle using matrix formulation

The core idea behind this formulation is the fact that every observable can be represented by a Hermitian matrix. For that purpose, the original work by Heisenberg [1] and Schrodinger [45] will be discussed that has been translated to English in [2] on the 70th anniversary of Uncertainty Relation.

The expectation value of an observable represented by operator \hat{A} can be given as

$$\bar{A} = \int \Psi^* A \Psi dx \tag{4}$$

Where, Ψ represents a complex wave function representing the state where the measurement will be done and Ψ^* is the complex conjugate of that function.

The operator \hat{A} , being an observable must give real eigen values which essentially makes it a Hermitian operator.

Now, the fluctuation in the value of \hat{A} can be given as

$$\Delta A = \sqrt{(A - \bar{A})^2} = \sqrt{A^2 - (\bar{A})^2} \tag{5}$$

Similarly, for another observable represented by a Hermitian operator \hat{B} , the fluctuation is

$$\Delta B = \sqrt{(B - \bar{B})^2} = \sqrt{B^2 - (\bar{B})^2} \tag{6}$$

Mathematical tool 1: The product of two Hermitian operators can be written as a combination of two parts [2]

$$AB = \frac{AB + BA}{2} + \frac{AB - BA}{2} \tag{7}$$

Mathematical tool 2: In order to prove the principle we need the help of Schwartz Inequality

$$\int m^* m dx \int n^* n dx \geq \left| \int m^* n dx \right|^2 \tag{8}$$

Putting $m = B\Psi$ and $n = A^*\Psi^*$

$$\int \Psi^* B B \Psi dx \int \Psi^* A A \Psi dx \geq \left| \int \Psi^* A B \Psi dx \right|^2 \tag{9}$$

$$\bar{A}^2 \cdot \bar{B}^2 \geq \left(\frac{AB + BA}{2} \right)^2 + \left| \frac{AB - BA}{2} \right|^2 \tag{10}$$

In order to get the more familiar form, we can write $A - aI$ and $B - bI$ in places of A and B .

Here a and b are real constants which are being multiplied to the identity operator I . The above inequality holds true for any wave function and any pair of real constants. Therefore, we can freely choose the value of a and b to be \bar{A} and \bar{B} : $A \rightarrow A - \bar{A}$ and $B \rightarrow B - \bar{B}$

On putting them on the above equation,

$$(\Delta A)^2 (\Delta B)^2 \geq \left(\frac{AB + BA}{2} - \bar{A}\bar{B} \right)^2 + \left| \frac{AB - BA}{2} \right|^2 \tag{11}$$

The cancellation of some terms is due to the fact that the expectation value of the operator is a number and taking another expectation value on the already expected number does not change anything. This inequality is the Schrodinger's uncertainty relation.

One can go to the Heisenberg's inequality by choosing the two operators \hat{A} and \hat{B} to be canonically conjugate. The final form of the inequality becomes

$$\Delta A \cdot \Delta B \geq \frac{\hbar}{2} \tag{12}$$

Modified form of HUP was also proposed by H.P. Robertson [9] in 1929 known as the Heisenberg-Robertson inequality:

$$\Delta A^2 \Delta B^2 \geq \frac{1}{4} |\langle [A, B] \rangle|^2 \tag{13}$$

It is noteworthy that the first term on the right hand side of equation (11) is the additional term to the Heisenberg-Robertson inequality. The term $\left(\frac{AB+BA}{2} - \bar{A}\bar{B} \right)$ is known as the co-variance. Schrodinger's equation is more general in the sense that it reduces to the other known forms for vanishing co-variance [44].

In terms of anti-commutator, Schrodinger's equation takes the form:

$$\Delta A^2 \Delta B^2 \geq \frac{1}{4} |\langle [A, B] \rangle|^2 + \frac{1}{4} |\langle \{A, B\} \rangle - 2\langle A \rangle \langle B \rangle|^2 \tag{14}$$

Where $\{ \}$ denotes the anti-commutator of the two operators.

2.2 Uncertainty relation using Fourier analysis

We will now examine the relationship between the probability density functions for position and momentum. For that we will use $f(x)$, a function of position and, $g(s)$, for the Fourier transform.

The fluctuation in the function f is given by [5]

$$\Delta f = \frac{\int x^2 |f(x)|^2 dx}{\int |f(x)|^2 dx} \tag{15}$$

From the paper of W. Du [5];

For f such that f, xf , and sg in L^2 ,

$$\Delta f \Delta g \geq \frac{1}{4} \tag{16}$$

Starting with Schrodinger equation

$$i\hbar \frac{\delta \chi}{\delta t} = \frac{-\hbar^2 \delta \chi^2}{2m \delta^2 x} + v(x)\chi(x, t) \tag{17}$$

Let us assume that the particle is in free space, $v(x) = 0$, then solution of the above equation is given by

$$\chi(x, t) = \frac{1}{\sqrt{2\pi\hbar}} \int \theta(p) e^{\frac{i(px-Et)}{\hbar}} dp \tag{18}$$

Where, χ, θ are in L^2 , and $\langle \chi | \chi \rangle = 1$ and $\langle \theta | \theta \rangle = 1$. We would like to define

$$\tilde{\chi}(p) = \frac{1}{2\pi\hbar} \hat{\chi}(p/\hbar) \tag{19}$$

Using inversion theorem, we get

$$\chi(x) = \frac{1}{2\pi} \int \chi\left(\frac{p}{\hbar}\right) e^{-ti\theta} d\left(\frac{p}{\hbar}\right) = \frac{1}{2\pi\hbar} \int \sqrt{2\pi\hbar} \tilde{\chi}(p) e^{-\frac{ip}{\hbar}x} dp \tag{20}$$

Now, comparing equation (20) with (18), We can take

$$\tilde{\chi}(p) = \theta(p) e^{-iEt/\hbar} \tag{21}$$

We get

$$|\theta(p)|^2 = |\tilde{\chi}(p)|^2 = \frac{1}{2\pi\hbar} \left| \tilde{\chi}\left(\frac{p}{\hbar}\right) \right|^2 \tag{22}$$

Substituting equation (22) in equation (15)

$$\begin{aligned} \Delta \hat{\chi} &= \frac{\int \left(\frac{p}{\hbar}\right)^2 \left| \tilde{\chi}\left(\frac{p}{\hbar}\right) \right|^2 d\left(\frac{p}{\hbar}\right)}{\int \left| \tilde{\chi}\left(\frac{p}{\hbar}\right) \right|^2 d\left(\frac{p}{\hbar}\right)} \\ &= \frac{2\pi\hbar}{\hbar} \frac{\int \left(\frac{p}{\hbar}\right)^2 |\theta(p)|^2 dp}{\int |\theta(p)|^2 dp} \\ &= \frac{1}{\hbar^2} \Delta \theta \end{aligned} \tag{23}$$

Since,

$$\Delta \chi \Delta \hat{\chi} \geq \frac{1}{4} \tag{24}$$

Therefore

$$\Delta \chi \Delta \theta \geq \frac{\hbar^2}{4} \tag{25}$$

It can be shown that $|\chi(x, t)|^2$ and $|\theta(p)|^2$ can be interpreted as probability density function for position and momentum respectively. Equation (25) is the required uncertainty relation which implies that the more precisely we measure the position we get more uncertain about the momentum.

It can be interpreted as follows: We know that the De-Broglie wavelength of a particle is related to the momentum of a particle. When we try to have a minimum uncertainty in the measurement of position, we need to localize the wave. For that purpose; according to Fourier series theory, more number of waves having different well-defined wavelengths have to be added. This in turn affects the certainty in momentum measurement which is related to the wavelength.

From the above two discussions it can be realized that the mathematics strongly imply the inevitable existence of the uncertainty principle. Therefore, one must be very careful while modifying the theory.

Some other mathematical formulations proving the uncertainty relation are given in Table 1.

Table 1: A variety of approaches used to treat HUP across literature

Methods	References
Matrix Formulation	[2,9,45]
Fourier analysis	[5]
Fractional Fourier analysis	[49,50]
Stochastic phase space method	[53]
Li-Ostojca-Starzewski fractional gradient operator method	[51]
Measurement operator formalism	[52]
Entropic treatment	[35,46,47]

2.3 Example: Testing HUP on a coherent state

The lower bound of the uncertainty principle gives rise to the minimum uncertainty states. These states can be attained by doing the experiments in the most careful way such that all the noises are ruled out. But it is not necessary that in a minimum uncertainty state, the uncertainties in the two quadratures are equal. A minimum uncertainty state having the same amount of uncertainty in the two quadratures is known as coherent state [4, 12].

As an example, we can test the Uncertainty Principle on a coherent state by finding the fluctuations in the two quadratures. For that we need to calculate the variance of each quadrature in a coherent state.

As a superposition of number states, a coherent state can be written as [4, 11].

$$\langle \alpha | = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle \tag{26}$$

Where $|n\rangle$ represents Folk number state and α is a complex number that follows

$$a < \alpha | = \alpha < \alpha | \quad (27)$$

Where a is the annihilation operator.

We can calculate the variance in position (q) and momentum (p) using the example of harmonic oscillator. For a harmonic oscillator, it is found that [1],

$$(\Delta q)^2 = \frac{\hbar}{2\omega} \quad (28)$$

And

$$(\Delta p)^2 = \frac{\hbar\omega}{2} \quad (29)$$

Such that

$$\Delta q \cdot \Delta p = \frac{\hbar}{2} \quad (30)$$

Therefore, the state is a minimum uncertainty state which obeys Heisenberg's Uncertainty principle for position and momentum.

It is possible to write the annihilation operator as a linear combination of two Hermitian operators ϵ_1 and ϵ_2

$$a = \frac{\epsilon_1 + i\epsilon_2}{2} \quad (31)$$

The two Hermitian operators follows the commutation relation [4]

$$[\epsilon_1, \epsilon_2] = 2i \quad (32)$$

Using this relation in Robertson's equation, the corresponding relation in terms of uncertainties becomes

$$\Delta\epsilon_1 \Delta\epsilon_2 \geq 1 \quad (33)$$

The product of the uncertainties $\Delta\epsilon_1$ and $\Delta\epsilon_2$ is minimum for a coherent state. Also, the uncertainties in the two quadratures are related as (which can be calculated using the state in equation (26))

$$\Delta\epsilon_1 = \Delta\epsilon_2 \quad (34)$$

As the uncertainty is minimum in a coherent state it is a close approximation of the classical light.

Any measurement in the two quadratures ϵ_1 and ϵ_2 will reveal the same amount of accuracy. Therefore lights where the uncertainty in one quadrature is lesser (squeezed light) can unveil information with more precision [13, 14].

3. Variants of uncertainty principle and their applications

When it came to non-classical phenomena like squeezing, the known form of the uncertainty principle performed flawlessly, even if there was a possibility that it was broken. But over time, it became clear that there are some instances where the uncertainty principle needs to be changed or reexamined. The existing expressions discussed till now were non-relativistic and without the consideration of its behaviour at a minimal length scale [16-25] which essentially originates from the theories of Quantum Gravity. We present the modified forms of the Uncertainty Relation in the domains of Quantum gravity and relativity in sections 4.1 and 4.2 respectively. Besides, the possibility to express the uncertainty relation in terms of entropy has become a great tool in Information theory [section

4.3]. We also present a glimpse of some other variants of the Uncertainty Principle in section 4.4. Some potential applications developed in the recent time are discussed in each case.

3.1 Generalised uncertainty principle

The known form of the Heisenberg's uncertainty principle is a consequence of the canonical relation between position and momentum operators. Various Theories of Quantum Gravity suggests a modified canonical relation due to the existence of a minimum length scale [16-25]. Commutation relation of this new form gives rise to uncertainty relations which are known as Generalised Uncertainty Principle. In the later phases, GUP has developed great applications in statistical mechanics [59-77], cosmology [78-100] etc.

While considering the theory of GUP, the paper by Pasquale Bosso et. al. [33] derives attention towards heuristic Vs explicit approaches towards theory formulation. Without denying the former, it can be proved that sometimes the two approaches produce significantly different outcomes. From a phenomenological point of view, several constrains on the model parameters of GUP has been introduced [33, 40-43]. The model parameter β of GUP has been tested by several experiments and accordingly prescribes different bounds. The strictest bound comes from the Lorentz-Invariance-Violations (LIV) in hydrogen atom experiment [32] which is

$$\beta > 10^{30}$$

The range of values of β starting from 10^{16} in Time of Flight measurements experiments [42] to 10^{93} in gravitational bar detectors experiment [43] questions the efforts of different experiments.

There exist other kinds of GUPs which rely on the theory of doubly special relativity [101-103]. A doubly special relativistic GUP is quite different from the more familiar GUP. The former gives an uncertainty relation which predicts both maximum momentum and minimal length. Below, we mention the uncertainty relation that has been obtained from GUP* [102, 103].

HUP*

$$\Delta x \cdot \Delta p \sim \frac{\hbar}{2} \left[-\beta \Delta p + \frac{1}{1-\beta \Delta p} \right]$$

$$\text{Where, } \beta = \frac{1}{M_p c}, M_p = \sqrt{\frac{\hbar c}{G}} = \text{Planck's Mass}$$

And the related commutation relation,

$$[x, p] = \frac{i\hbar}{1-\beta p}$$

It can be realised that the relation naturally predicts a maximum momentum $p_{max} = \frac{1}{\beta}$

This doubly special relativistic GUP has been widely used in different quantum gravitational theories. In a recent work by Daniele Gregoris et. al. such a theory proposed in [103] has been used to derive the Chandrashekhar's mass for white dwarfs which is consistent with Chandrashekhar's point of view which emphasized that the mass limit actually arises for a vanishing size of white dwarf.

It's worthy to mention other astrophysical implications of this modified Heisenberg principle. For example, it affects the Hawking evaporation of black holes in such a way that remnants may exist [104]. The theory predicts viable candidate for dark matter [105]. It is also claimed that the baryon asymmetry may also due to GUP included effects [106].

In literature studies can be found to probe GUP using opto-mechanical system [107-109]. Among various attempts to probe GUP, the recent work by Parth *et al* [110]. is worth mentioning. The authors carry out a phenomenological approach to test the GUP through quantum noise using observations from several recent experiments like position noise spectrum of advanced LIGO to constraint the modified commutator. The noisy behaviour of an opto-mechanical system has been studied using modified commutator. From these discussions it can be realised that probing the mechanical noise of harmonic oscillator with high precision, as in recent experiments, is a viable technique to contain small modifications of Heisenberg's Uncertainty Principle found in the models of Quantum Gravity.

The modified Quantum mechanics has its implications in the neutrino physics also as it is shown by Banerjee *et al* In [111]. In this work the authors did a thorough investigation of the neutrino de-coherence using Quantum Gravity prescribed GUP. As the neutrinos are almost non-interactive, they can travel a large distance without losing their coherence which means the wavefunctions of two identical neutrinos moving along the same trajectory and equal energy will evolve identically. But, this coherence can be lost when the neutrinos interact with the environment over large distance.

A modified equation for the evolution of a quantum state results from the GUP's modifications to the canonical equations. The neutrino state's evolution is determined by this. In the end, these changes result in decoherence [111]. Shows that the LQGUP model predicts a decoherence in the neutrino oscillation with de-coherence parameter having a fourth order power law dependence on energy.

3.2 Relativity and Uncertainty principle

The original Heisenberg Uncertainty principle is not consistent with the theory of special relativity. Although some theories have been already proposed in the ultra-high velocity regime [112], proposes a theory that can be experimentally probed in an intermediate energy range. The relativistic correction to the Heisenberg's Principle obtained from their work is

$$\delta x^2 \cdot \delta p^2 \geq \frac{1}{4} \left(1 + \frac{3}{2} \frac{\delta p^2}{m^2 c^2} \right)$$

The key conclusion, according to the authors, is a leading order special relativistic correction which is the most reliable and appropriate for serving as a guide for experimental investigations of the interaction between the Heisenberg's principle and special relativity.

At a first instance the realistic corrections look very similar to the GUP corrections. For example, the relativistic correction to the Hamiltonian is given by $\frac{\lambda_c^2 k^4}{8m\hbar^2}$ and the GUP correction to the Hamiltonian is $\frac{\beta l^2 k^4}{m\hbar^2}$ which are quite similar. But, the Compton

wavelength λ_c and the minimal length l make the difference clear. While the former is particle specific and the later is a constant [27, 30, 31, 33]. Although, it doesn't confirm that the GUPs are non-relativistic. The unification of relativistic and GUP formalism is yet to be investigated thoroughly.

It is necessary to consider the relativistic case when we deal with microscopic quantum objects. There have been efforts to extend the scope of the uncertainty principle to the relativistic realm [26-31]. As an extension of the HUP to relativistic domain or Lorentz space time, one can write the modified commutation relation as [33]

$$[x^m, p_n] = i\hbar\delta_n^m \quad (35)$$

Although extension is simpler, it has some limitations. The extension can't be applied to a multi-particle scenario which is a possibility of the relativistic realm [48]. Again, defining a self-adjoint time operator is impossible in this formation [28, 29].

The question of relativistic extension is also applicable to the Generalised Uncertainty Principle.

The HUP is violated by Quantum Gravity theories such as String theory [113-115], Loop Quantum Gravity [116-119] and Doubly Special Relativistic theories [120] which all predict a minimum measurable length in spacetime. The minimum length is described by Quantum Gravity theories with the help of Generalised Uncertainty Principle (GUP). However, it violates Lorentz covariance at high energies since measured lengths are not Lorentz invariant quantities. Due to this difficulty, all the above-mentioned GUPs are non-relativistic. There have been efforts to apply GUP in relativistic and Quantum field theory [121-125], but the minimum length is still not Lorentz invariant. The work by [126, 127] suggests a Lorentz invariant GUP that includes a minimum length and reduces to its non-relativistic variants at low energies. The paper also applies the modified relation to compute GUP correction to Dirac equation, Schrödinger's equation and Klein-Gordon equation.

Another mentionable work by [128] take a different approach in the relativistic realm of Uncertainty Principle. This work explores relativistic adjustments to the uncertainty relation that are based on the curvature of background space-time. It generalizes earlier work and computes the standard deviation of the relativistic physical momentum operator for wave functions restricted to a geodesic ball on a space-like hypersurface by using the 3+1-splitting according to the ADM-formalism. Position uncertainty is measured by this operator's radius. The velocity of the particle, the non-vanishing time-space components of the space-time metric, and the Ricci scalar of the effective spatial metric determine the corrections to the uncertainty relation in flat space. It can be applied to both mass less and hefty particles.

3.3 Entropic uncertainty relation

Technological advancements & innovations motivated the scientific society to study the uncertainty principles from different perspectives. Information theory, which is the basis for communication technology, quantum cryptography, quantum metrology etc., provides a framework that can realise

the abstract notions of uncertainty and the most frequently used mathematical tool to express uncertainty in information theory is entropy [129]. It is even argued that entropy for uncertainty is the most natural choice [129].

The motivation to study entropic relations is not only confined to its applicability in the information theory. It is also argued that assigning standard deviation as a measure of uncertainty is sometimes questionable [130]. Moreover entropic uncertainty framework is blessed with the fact that it can be generalised to correlated systems quite easily [129].

One of the first such entropic relations given by Bialynicki - Birula & Mycielski [131] in terms of differential entropy “ h ” is

$$h(Q) + h(P) \geq \log(\epsilon\pi\hbar)$$

Here, Q & P are the position and momentum respectively.

The theory has been extended and improved by different authors that can be seen from [132-134]. Out of these myriad formulations, the most common and familiar relation was given by Maassen and Uffink [133] in terms of Shannon’s entropy H , which is

$$H(x) + H(z) \geq \log \frac{1}{c}$$

The term “ c ” represents the maximum overlapping between the two Eigen vectors of the observables x and z .

The entropic uncertainty relation finds its application in Quantum key distribution (QKD). Quantum key distribution is a scheme where two parties Alice and BOB agree on a shared key through a public channel. But a classical public channel is very sensitive to eavesdropping due to which quantum channels are preferred for such communications. The idea of QKD was first proposed by [135]. The uncertainty principle is mentioned in one way or another in all of the security arguments put out by different scientists [136, 137]. For QKD, entropic uncertainty relations have been used by [138, 139].

In literature, application of entropic uncertainty relations can be found in two party cryptography also [140, 141]. Unlike QKD, two party cryptography is a communication scheme between two mutually distrustful parties.

It is also used in quantum metrology [142, 143]. Quantum metrology is related to the limits on accuracy of measurement. Optical measurements (e.g. involving interferometers) require phase measurements where uncertainty relations involving phase play an important role. Hall *et al.* [142] applied entropic U.R and formulated a relation

$$H(N) + h(\theta) \geq \log 2\pi$$

Where, $\langle N \rangle$ is the mean photon number of the light that has been used for measurement.

Here, h is Shannon entropy which is related to $\delta\varphi$

And $H(N)$ is related to $\langle N+1 \rangle$

$\delta\varphi$ =R.M.S. deviation of the actual phase φ from the measured phase $\bar{\varphi}$

$\theta = \bar{\varphi} - \varphi$

Quantum information technologies depend on techniques for detecting entanglement, which is a vital resource in quantum information processing. The procedure of confirming that a source is generating entangled particles is known as entanglement witnessing. The field of entanglement witnessing is well developed and comprises various forms, with a primary

emphasis on entropic uncertainty relations. In the distant-laboratories paradigm, bipartite entanglement is discussed in [129]. In this scenario, two parties are able to take local measurements on their respective systems, but are unable to perform a global measurement on the system. Another paper [144] investigates lower constraints on entanglement measures based on entropic uncertainty relations, which are important for basic concerns in condensed-matter physics and quantum statistical mechanics, as well as for quantum simulation experiments.

Another work [145] compares experimental findings with predictions to analyze entropic uncertainty relations pertinent to neutrino-flavor states. Ensembles of accelerator and reactor neutrinos from several sources, including Daya Bay and MINOS collaboration data, are included in the analysis. The findings indicate that when energy increases, non-monotonic evolutions occur in entropy-based uncertainty conditions. Systemic quantum-ness is quantified by quantum correlation, which also establishes an inherent connection between EUR and quantum correlation. By detecting the entanglement of neutrino-flavor states, EUR highlights how neutrino oscillations may be used to process quantum information in weak interaction processes.

Apart from these areas, entropic uncertainty relations are also useful in quantum coding [146, 147], coherence [148-150] etc.

Furthermore, in literature, there can be found another set of theories which are an extended version of the known form of HUPs without considering relativistic or minimal length criteria [10, 34-39]. It can be shown that the traditional form of the principle is a leading order approximation of such generalized extended principles [10]. The higher order dependence of observables in these theories unveils new and interesting information.

Apart from the traditional form, variance based uncertainty relations have been studied in detail [34, 36, 10]. But, theories on uncertainty relations with higher order moments are still in a nascent stage [10].

Some papers try to give stronger form of uncertainty relation in terms of sum of the square of the variances instead of their product [34]. These works are important as they consider the case of incompatible observables also.

Some other forms are being prescribed for multiple observables which are applicable to both pure and mixed systems etc [36].

4. A brief look at experimental progress to demonstrate the uncertainty principle

To give a perspective on the experimental works that has been done on the uncertainty principle, we want to mention some of the previous works.

The findings of an experiment measuring the longitudinal coherence length of a thermal neutron beam using neutron interferometry were published in 1983. It was published by H. Kaiser, S. A. Werner, and E. A. George [54]. Where they performed an experiment using a three-crystal silicon interferometer of the type developed by Bonse and Hart for x rays and later shown to work for neutrons by Rauch, Treimer and Bonse. They placed a polished, pure aluminium slab in one

leg of the interferometer as a result the wave packet traversing this path experienced an optical potential in interacting with aluminum nuclei and the wave packet was delayed compared to the wave packet traversing the other path. They used 1.268 Å neutrons in this experiment which gave a spatial shift of 54 Å. They measured the contrast by rotating a phase shifting “flag” of certain thickness and studied the change of contrast as a function of the relative spatial shift which is the convolution of the wave packets integrated over the detector and averaged over time. They calculated the width of the amplitude function and the spatial width of the wave packets which is in a good agreement with the uncertainty relation. The results of this experiments were interpreted as the verification of the uncertainty principle for neutrons in the longitudinal direction.

In the works of Olaf Nairz *et al.* in 2002 [55], they demonstrated the Heisenberg uncertainty principle for the fullerene molecule C70 at a temperature of 900 K. They used sublimation of fullerene powder at 900 K to produce a thermal beam C70. Two piezo controlled slit, the first slit with a fixed width of 10 μm, while the second slit with variable width placed 113cm away from the first slit, are used to collimate the molecular beam. The beam is narrowed by the first slit and diffracted by the second slit. Three different methods were used to get information about the slit width. The first one is from piezo voltage, second from the reading of a strain gauge mounted to the slits and finally from the total number of molecules passing through the slits. Then, they calculated the momentum spread from full width half maxima of the detected molecular beam in the detection plate which is placed 133 cm away from the second slit. The observed distribution function is the convolution of the detector resolution function and real molecular beam profile. To study the uncertainty relation, they varied the width of the second slit from 20 μm down to 50 nm and observed the beam width. They compared the experimental data with the predicted curve and it was found to be in good agreement. So, they concluded the validity of the Heisenberg uncertainty principle for the fullerene C70. These studies provide more evidence that even bigger molecules and clusters may exhibit quantum interference events.

In a 2010 paper of Giorgio Matteucci *et al.* [56], diffraction of electron was used to examine the uncertainty principle. The study used a typical electron microscope equipped with a high coherent source and a readily accessible thin foil aperture to present an electron diffraction experiment from a circular aperture. They used a FEI Tecnai F20 electron microscope with a schottky field emission source operating at 200 kV accelerating voltage. When electrons are emitted from the source it goes through a condenser lens system which forms a parallel electron beam on the circular aperture. Then they used a projector system to magnify the diffraction pattern by about 50 times. They analyzed the intensity distribution as a function of distance from the center of the aperture and found that to be coinciding with the quantum theoretical predictions i.e., the wave-like behavior of the electrons. However, the heights of maxima and depth of minima from the experimental data are found to be systematically higher than the theoretical curve.

This is due to the residual thermal noise and high intensity of central maxima. They reduced the thermally generated electrons by using an electronic Peltier device. It was pointed out that when we consider an electron as a particle and it crosses the circular slit, its distance from the optical axis is undetermined by the slit radius. The diffraction pattern formed by a number of electrons on the screen is due to the spreading of the momentum when the electrons crosses the aperture. They calculated the spread in position and momentum and found that the result was consistent with the uncertainty principle.

In 2011, a paper was published by D Nikolic and Lj Nestic [57], in which the uncertainty principle was verified by using diffraction of light waves. They used a single slit diffraction of a laser for measuring the angular width of zero order maxima and obtained the corresponding wave number uncertainty. With a simple experimental setup, they measured the positions of diffraction minima for Fraunhofer diffraction on the slit. They used two blades mounted parallel to the carrier in the vertical plane. These were well positioned and fixed so that the fine tuning of the width of the slit possible. They used slit width between 0.05-0.65 mm. They used a photodetector to record the diffraction fringes. A He-Ne laser of power 2mW with wavelength 632.8nm was used to take measurements with 13 different slit widths. The uncertainty in position is equal to the width of the slit and it was directly measured by a thin leaf gauge. For uncertainty in wave number, they approached with an indirect method of measurement. They compared the measured values and found out that the results were in accordance with the uncertainty principle. They verified the position wave number uncertainty relation rather than position momentum uncertainty relation because they could not find a way to take photon momentum as an input parameter.

In another paper [58], the entropic uncertainty relation discussed in section 4.3 is experimentally being investigated and its application in observing entanglement is proposed.

5. Discussion and Conclusion

It is important to conclude that our discussions emphasized the fundamentals involved in the Uncertainty Principle, its variants and their applications in recent times. The intention of this work was to provide the facts in one place which deny the universality of the most familiar form of the Heisenberg's Uncertainty Principle in terms of position and momentum.

First, we presented a background of the uncertainty Principle and thoroughly discussed two mathematical approaches to prove the relation.

We noted that not all situations can be viewed as universally related by the well-known Heisenberg's Uncertainty Principle. In this regard it can be mentioned that Generalized Uncertainty Principle (GUP) formalism [section 4.1], which has its origin in the theories of Quantum Gravity is primarily related to Plank's length or minimal length approach. Individually, this approach is correct and valid. On the other hand, as discussed in section 4.2, relativistic approach is equally valid in its own domain. Although both the GUP and relativistic corrections look similar, it has been discussed where they differ. But that merely does not defy the possibility of the relativistic extension

of GUP. To realize the fundamental aspects of Quantum physics, conversion of both the approaches to a particular point is highly essential. We can desire the unification of these two approaches for useful conceptual frameworks. The no. of literatures in regard of relativistic extension of GUP is very few.

It has also been observed that in some literatures [32, 42, 43] the value of phenomenological constraint β of GUP models has a very wide range of variation, which has already been mentioned in section 4.1. For obtaining a universal value of β the effort on experiments should be stressed more or theoretical modifications may be needed for obtaining the same. It demands more investigations to illuminate the problem.

We tried to shed a light on the entropic approach to Uncertainty Principle too. It is argued for several times that the assignment of uncertainty with entropy is the most natural choice. Technological advancements in communication technology, quantum cryptography, quantum metrology which are the by-products of Quantum Information Theory pose a greater demand for entropic uncertainty principle.

One more observation has been found that most of the experiments relating to HUP are interferometry based. A very few reports can be found where other methods have been explicitly used. As a suggestive measure we can propose to investigate for other methods other than interferometry. The present report is a rich and ongoing area of research with the potential to impact a wide range of fields of quantum technology. As quantum technologies continue to advance, our understanding in uncertainty relations will play a vital role in shaping information processing in quantum systems and development of quantum communications.

Data availability statement

No new data were created or analyzed in this study.

Statements and declarations

We would like to mention that there is no conflict of interest among the authors regarding the publication of the work. This includes financial interest, relationships, affiliations or other circumstances that might be perceived as influencing the objectivity or integrity of the research conducted and the findings presented.

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