# On Finding Integer Solutions to Non-homogeneous Ternary Cubic Equation $[x^2 + xy + y^2 = (m^2 + 3n^2)z^3]$

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**Abstract**

The purpose of this paper is to obtain different sets of non-zero distinct integral solutions of ternary non-homogeneous cubic Diophantine equation $x^2 + xy + y^2 = (m^2 + 3n^2)z^3$. A few properties of interest are presented.

**Keywords:** cubic with three unknowns, non-homogeneous cubic, integer solutions

## Introduction

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, cubic diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-4]. In this context, one may refer [5-26] for various problems on the cubic diophantine equations with three variables, where, in each of the problems, different sets of non-zero integer solutions are obtained. However, often we come across homogeneous and non-homogeneous cubic equations and as such one may require its integral solution in its most general form. It is towards this end, this paper concerns with the problem of determining a general form of non-trivial integral solutions of the non-homogeneous cubic equation with three unknowns given by $x^2 + xy + y^2 = (m^2 + 3n^2)z^3$.

## Notations

$T_n$ - Triangular number of rank n

$Ob_n$ - Oblong number of rank n

$Th_n$ - Tetrahedral number of rank n

$PP_n$ - Pentagonal Pyramidal number of rank n

## Method of analysis

The ternary cubic equation under consideration is

$$x^2 + xy + y^2 = (m^2 + 3n^2)z^3 \tag{1}$$

**Different patterns of integral solutions of (1) are analyzed below:**

**Pattern I**

Substitution of

$$x = u + v, y = u - v \tag{2}$$

in (1), reduces it to

$$3u^2 + v^2 = (m^2 + 3n^2)z^3 \tag{3}$$

which can be written as

$$(v + i\sqrt{3}u)(v - i\sqrt{3}u) = (m + i\sqrt{3}n)(m - i\sqrt{3}n)(a + i\sqrt{3}b)^3(a - i\sqrt{3}b)^3$$

Where,

$$z = a^2 + 3b^2 \tag{4}$$

Assuming

$$\left(v + i\sqrt{3}u\right) = (m + i\sqrt{3}n)(a + i\sqrt{3}b)^3$$

and equating the real and imaginary parts, the values of u, v are obtained. In view of (2), we get the solutions of (1) as

$$\left.\begin{aligned} x &= m(a^3 - 9ab^2 + 3a^2b - 3b^3) + n(a^3 - 9ab^2 - 9a^2b + 9b^3) \\ y &= m(9ab^2 - a^3 + 3a^2b - 3b^3) + n(a^3 - 9ab^2 + 9a^2b - 9b^3) \end{aligned}\right\} \tag{5}$$

along with (4).

To analyze the nature of solutions of (1), one has to go in for particular values of $m$ and $n$

We present below the integral solutions of (1) for the choices of $m$ and $n$ given by

i. $m = 1 \ \ n = 0$

ii. $m = 0 \ \ n = 1$

**Choice I**

Taking $m = 1, n = 0$ in (1) & (5), we get the ternary cubic diophantine equation to be solved is

$$x^2 + xy + y^2 = z^3 \tag{6}$$

and the corresponding solutions are

$$\left.\begin{aligned} x &= (a^3 - 9ab^2 + 3a^2b - 3b^3) \\ y &= (9ab^2 - a^3 + 3a^2b - 3b^3) \end{aligned}\right\}$$

along with (4).

**Properties**

1. $\frac{a(x+y)}{6}$ represents a Nasty number, when $a = 2(r^4 + s^4), b = 4r^2s^2$

2. $\frac{(x-y)}{a}$ represents a Nasty number, when $a = 3r^2 + 9s^2; b = r^2 - 3s^2$
3. $x + y$ represents a Nasty number for the following choices of $a$ and $b$.

a) $a = (2^{4\alpha-2} + 1)q^2; b = 2^{2\alpha}q^2$

b) $a = 4k^4 + 8k^3 + 6k^2 + 2k,$
$b = 4k^4 + 8k^3 + 4k^2 (or) - 4k^4 - 8k^3 - 8k^2 - 4k - 1$

c) $a = 2r^4 + 2s^4; b = 4r^2s^2$

4. Each of the expressions $\frac{x^2-y^2}{6a(a+3b)}, \frac{x^2-y^2}{6a(a-3b)}$ $\frac{3b(x-y)}{2}$ is a Nasty number.
5. $9(6bz - x - y), x - y + 6az$ are cubical integers.
6. $z$ is a square when $a = 3r^2 - s^2\ (or)\ r^2 - 3s^2, b = 2rs$.
7. $\frac{x^2-y^2}{12ab}$ is written as the product or four numbers in Arithmetic Progression.
8. When $a = 5r^2 + s^2 - 4rs,\ (or)\ r^2 + 5s^2 - 4rs;\ b = 2rs$ $\frac{y+az}{3b}$ is a perfect square.
9. When $a = 5r^2 + s^2 + 4rs$ (or) $r^2 + 5s^2 + 4rs; b = 2rs$, $\frac{x-az}{3b}$ is a perfect square.
10. $x + y + z$ is a perfect square when

a) $a = 4b, b = 90u^2 + 34u + 3$

b) $a = 2b, b = 18u^2 + 10u + 1$

11. $\frac{x+y+z}{3}$ is a perfect square when $a = \pm 6T_\alpha, b = 2T_\alpha$.

It is worth mentioning that (6) is also solved through a different approach as follows:

On completing the squares, (6) is written as

$$(2x + y)^2 + 3y^2 = 4z^3 \qquad (7)$$

The substitution of the linear transformations

$$x = 2X, y = 2Y \qquad (8)$$

in (7) leads to

$$(2X + Y)^2 + 3Y^2 = z^3 \qquad (9)$$

Using (4) in (9), it is written in the factorized form as

$$\left((2X+Y)+i\sqrt{3}Y\right)\left((2X+Y)-i\sqrt{3}Y\right) = (a+i\sqrt{3}b)^3(a-i\sqrt{3}b)^3$$

Defining

$$\left((2X+Y)+i\sqrt{3}Y\right) = (a+i\sqrt{3}b)^3$$

and equating the rational and irrational parts, we get

$$2X + Y = a^3 - 9ab^2 \qquad (10)$$

$$Y = 3a^2b - 3b^3 \qquad (11)$$

Solving (10) and (11) for $X$ and employing (8), we have

$$x = a^3 - 9ab^2 - 3a^2b + 3b^3 \qquad (12)$$

$$y = 6a^2b - 6b^3 \qquad (13)$$

Thus, (4), (12) and (13) represent the non-trivial integral solutions of (6) provided $a \neq b$.

We preset below a few interesting relations among the solutions:

1. Each of the expressions $\frac{3b}{2}(2x + y), (a > 3b), \frac{ay}{6}$ represents a Nasty number.
2. If $b = 1$, then $ay = 36\,Th_{(a-1)}$
3. If $a = 2n + 1, b = 1$, then $y = 48, T_n = 96\,Ob_n$
4. If $a = -b$, then $x$ represents a cubical integer. Also, if $a = -3B, b = -B$then $9x$ is a cubical integer.

Representing the solutions $x, y, z$ of (12), (13) & (4) by the notations $x(a,b), y(a,b), z(a,b)$ respectively, the following relations are observed:

5. $x(a,b) = -x(-a,-b)$
6. $x(3b,b) = x(-3b,b)$
7. $x(-3b,-b) = 3x(-b,b)$
8. $a[x(-a,-1),-x(-a,1)] = 36\ Th_{(a-1)}$
9. Each of the expressions represents a Nasty number

a) $2b[y(a+2b,b) - 2y(a+b,b) + y(a,b)]$

b) $2b[y(a+b,b) + y(a-b,b)]$

c) $a[y(a+b,b) - y(a-b,b)]$

d) $3[z(a+2b,b) - 2z(a+b,b) + z(a,b)]$

e) $z(a+2b,b) + 2z(a-b,b) - 3z(a,b)$

f) $6ab[z(a+b,b) - z(a-b,b)]$

g) $3ab[y^2(a+b,b) - y^2(a-b,b)]$

10. $6b[z(a + 2b, b) - 2z(a + b, b)] = y(a + 2b, b) - 2y(a + b, b) + y(a.b)$
11. $6[y^2(a + b, b) - y^2(a - b, b)]$ is a cubical integer.
12. $2[y^2(a + b, b) + y^2(a - b, b)] = 9(z + b^2)[z(a + b, b) - z(a - b, b)]^2$
13. $[y(a+1,1) - y(a,1)]^2 = 36[8\ T_a + 1]$
14. $y(1+2b,b) - y(1,b) = 48\ PP_b$
15. $y(1+2b,b) - y(1+b,b) + 6b^2 = 36\ PP_b$
16. Each of the following expressions is written as the difference of two squares

a) $z(a+b,b) - z(a-b,b)$

b) $z(a+b,b) - z(a,b)$

c) $z(a,b) - z(a-b,b)$

d) $\frac{y(a+1) - y(a,1)}{6}$

**Choice II**

Taking $m = 0, n = 1$ in (1) & (5), we get the ternary cubic diophantine equation to be solved is

$$x^2 + xy + y^2 = 3z^3 \qquad (14)$$

and the corresponding solutions are

$$\left.\begin{array}{l} x = (a^3 - 9ab^2 - 9a^2b + 9b^3) \\ y = (a^3 - 9ab^2 + 9a^2b - 9b^3) \end{array}\right\}$$

along with (4).

It is worth mentioning that (14) is also solved through a different approach as follows:

On completing the squares, (14) is written as

$$(2x + y)^2 + 3y^2 = 12z^3 \qquad (15)$$

Using (4) in (15), it is written in the factorized form as

$$\left((2x+y)+i\sqrt{3}y\right)\left((2x+y)-i\sqrt{3}y\right) = (a+i\sqrt{3}b)^3(a-i\sqrt{3}b)^3(3+i\sqrt{3})(3-i\sqrt{3})$$

Defining

$$\left((2x+y)+i\sqrt{3}y\right) = (3+i\sqrt{3})\left((a^3-9ab^2)+i\sqrt{3}(3a^2b-3b^3)\right)$$

and equating the rational and irrational parts, we get

$$2x+y=3a^3-27ab^2-9a^2b+9b^3 \quad (16)$$

$$y=9a^2b-9b^3+a^3-9ab^2 \quad (17)$$

Solving (16) and (17), we get

$$x=a^3-9ab^2-9a^2b+9b^3 \quad (18)$$

Thus (4), (17) and (18) give the integral solutions to (14).

The following properties are satisfied by the above solutions

1. $3b(x+y)$ is two times a Nasty number.
2. $\frac{a(y-x)}{18}$ is a Nasty number
3. When $b=1, a(y-x)=108\,Th_{a-1}$
4. $z$ is a perfect square when $b=2rs, a=3r^2-s^2$(or) $a=r^2-3s^2$
5. When $a=9(p^2+q^2)^2, b=3(6p^2q^2-p^4-q^4)$, $\frac{x+y}{3}$ is a Nasty number.
6. When $a=9(R^2+S^2), b=3(R^2-S^2)$, $\frac{x+y}{2}$ is written as the sum of two squares.
7. When $a=2(p^4+q^4), b=4p^2q^2$, $\frac{y-x}{3}$ is a Nasty number.
8. Each of the expressions represents $x+y+6az=8a^3$, $3(x-y+18bz)=216b^3$ a cubical integer
9. Denoting the solutions of $x,y,z$ of (6.30) by the notations $x(a,b), y(a,b), z(a,b)$ respectively, the following relations are observed

a) $x(a,b)+x(-a,-b)=0$
b) $a[x(-a,-b)+y(a,b)]=18$ (Area of the Pythagorean triangle with generators $(a,b)(a.>b)$)
c) $a[x(-a,-1)+y(a,1)]=108Th_{(a-1)}$
d) $y(3b,b)+x(3b,b)=0$
e) $y(a,-b)=x(a,b)$
f) $x(-a,-1)+2PP_a-18T_a+9$ is a perfect square.
g) $y(1,b)+18PP_b-1\equiv 0(mod\,9)$
h) $x(a,1)+y(a,1)-4PP_a+4T_a\equiv 0(mod\,16)$
i) If $a>3b, \frac{3b}{2}[x(a,b)+y(a,b)]$ is the area of a Pythagorean triangle whose generators are $a, 3b$
j) If $ab$ is a perfect square, then $3[x(a,b-1)+y(a,b-1)-x(a,b+1)-y(a,b+1)]$ is a Nasty number
k) $2a^2z-ax(a,b)-ay(a,b)$ is a Nasty number
l) $[x(a,b+1)-x(a,b)+y(a,b+1)-y(a,b)]^2=324a^2[8T_b+1]$
m) $3y(3b,b)$ is a cubical integer
n) If $a=b^2$, $3[x(a,b-1)+y(a,b-1)-x(a,b+1)-y(a,b+1)]$ is a cubical integer.
o) $x(a,b)+y(a,b)+6az(a,b)$ is a cubical integer
p) $18bz-x(9b,b)-y(9b,b)$ is a cubical integer

## Conclusion

In this paper, we have made an attempt to find non-zero distinct integer solutions to the non-homogeneous cubic equation with three unknowns given by $x^2+xy+y^2=(m^2+3n^2)z^3$. To conclude, one may search for other choices of general form of integer solutions to the cubic equation with three unknowns in title.

## References


1. Dickson LE. History of Theory of Numbers, Chelsea publishing company, New York, 1952, 2.
2. Carmichael RD. The Theory of Numbers and Diophantine Analysis, Dover Publications, New York, 1959.
3. Mordell LJ. Diophantine Equations, Academic press, London, 1969.
4. Telang SG. Number Theory, Tata Mcgrow Hill Publishing company, New Delhi, 1996.
5. Gopalan MA, Srividhya G. Integral solutions of ternary cubic diophantine equation $x^3+y^3=z^2$, Acta Ciencia Indica, 2011; 37(4):805-808.
6. Gopalan MA, Vidhyalakshmi S, Mallika S. On the ternary non-homogeneous Cubic equation $x^3+y^3-3(x+y)=2(3k^2-2)z^3$, Impact journal of science and Technology, 2013; 7(1):41-45.
7. Gopalan MA, Vidhyalakshmi S, Thiruniraiselvi N. On homogeneous cubic equation with three unknowns $x^2-y^2+z^2=2kxyz$, Bulletin of Mathematics and Statistics Research, 2013; 1(1):13-15.
8. Gopalan MA, Vidhyalakshmi S, Lakshmi K. Latice Points On The Non-homogeneous cubic equation $x^3+y^3+z^3+x+y+z=0$, Impact J. Sci. Tech, 2013; 7(1):21-25.
9. Gopalan MA, Vidhyalakshmi S, Lakshmi K. Latice Points on The Non-homogeneous cubic equation $x^3+y^3+z^3-(x+y+z)=0$, Impact J. Sci. Tech, 2013; 7(1):51-55.
10. Vidhyalakshmi S, Usharani TR, Gopalan MA. Integral Solutions of the Ternary cubic Equation $5(x^2+y^2)-9xy+x+y+1=35z^3$, International Journal of Research in Engineering and Technology, 2014; 3(11):449-452.
11. Gopalan MA, Vidhyalakshmi S, Mallika S. Integral solutions of $x^3+y^3+z^3=3xyz+14(x+y)w^3$, International Journal of Innovative Research and Review, 2014; 2(4):18-22.
12. Gopalan MA, Vidhyalakshmi S, Mallika S. Non-homogeneous cubic equation with three unknowns $3(x^2+y^2)-5xy+2(x+y)+4=27z^3$, International Journal of Engineering Science and Research Technology, 2014; 3(12):138-141.
13. Gopalan MA, Thiruniraiselvi N, Sridevi R. On the ternary cubic equation $5(x^2+y^2)-8xy=74(k^2+s^2)z^3$, International Journal of Multidisciplinary Research and Modern Engineering, 2015; 1(1):317-319.
14. Gopalan MA, Thiruniraiselvi N, Krithika V. On the ternary cubic diophantine equation $7x^2-4y^2=3z^3$, International Journal of Recent Scientific Research, 2015; 6(9):6197-6199.
15. Gopalan MA, Vidhyalakshmi S, Shanthi J, Maheswari J. On ternary cubic Diophantine equation $3(x^2+y^2)-5xy+x+y+1=12z^3$, IJAR, 2015; 1(8):209-212.
16. Janaki G, Saranya P. On the ternary Cubic diophantine equation $5(x^2+y^2)-6xy+4(x+y)+4=40z^3$, International Journal of Science and Research-online, 2016; 5(3):227-229.

17. Anbuselvi R, Kannan K. On Ternary cubic Diophantine equation $3(x^2+y^2)-5xy+x+y+1=15z^3$, International Journal of scientific Research, 2016; 5(9):369-375.
18. Vijayasankar A, Gopalan MA, Krithika V. On the ternary cubic Diophantine equation $2(x^2+y^2)-3xy=56z^3$, Worldwide Journal of Multidisciplinary Research and Development, 2017; 3(11):6-9.
19. Janaki G, Saranya C. Integral Solutions of The Ternary Cubic Equation $3(x^2+y^2)-4xy+2(x+y+1)=972z^3$, IRJET, 2017; 4(3):665-669.
20. Anbuselvi R, Nandhini R. Observations on the ternary cubic Diophantine equation $x^2+y^2-xy=52z^3$, International Journal of Scientific Development and Research, 2018; 3(8):223-225.
21. Gopalan MA, Sharadhakumar. On the non-homogeneous Ternary cubic equation $3(x^2+y^2)-5xy+x+y+1=111z^3$, International Journal of Engineering and technology, 2018; 4(5):105-107.
22. Gopalan MA, Sharadhakumar. On the non-homogeneous Ternary cubic equation $(x+y)^2-3xy=12z^3$, IJCESR, 2018; 5(1):68-70.
23. Vijayasankar A, Sharadha Kumar, Gopalan MA. On Non-Homogeneous Ternary Cubic Equation $\boldsymbol{x^3+y^3+x+y=2z(2z^2-\alpha^2+1)}$, International Journal of Research Publication and Reviews, 2021; 2(8):592-598.
24. Vidhyalakshmi S, Shanthi J, Hema K, Gopalan MA. Observation on the paper entitled Integral Solution of the homogeneous ternary cubic equation $x^3+y^3=52(x+y)z^2$, EPRA IJMR, 2022; 8(2):266-273.
25. Vidhyalakshmi S, Gopalan MA, General form of integral solutions to the ternary non-homogeneous cubic equation $y^2+Dx^2=\alpha z^3$, IJRPR, 2022; 3(9):1776-1781.
26. Vidhyalakshmi S, Gopalan MA, On Finding Integer Solutions to Non-homogeneous Ternary Cubic Equation $x^2+by^2=(m^2+bn^2)z^3$, IRJEdT, 2022; 4(10):22-29.