Education

Mathematical investigation: The role of eccentricity in distinguishing conic sections

Alaiza B. Dela Cruz[1], Reisy Joy D. J. Castro[1], Daniela Marie Y. Lapuz[1], Shyraine Leigh J. Liwag[1*], Mark Ren D. Villaflor[1], Titin Rahmiatin Rahim[2] and Usman[2]

[1] Nueva Ecija University of Science and Technology, San Isidro, Nueva Ecija, Philippines

[2] Muhammadiyah University of Kendari, Kendari, Sulawesi Tenggara, Indonesia

*Corresponding author: Shyraine Leigh J. Liwag

Received 21 May 2026; Accepted 2 Jul 2026; Published 14 Jul 2026

DOI: https://doi.org/10.64171/JAES.6.4.12-16

Abstract

This investigation examines how eccentricity (e) distinguishes conic sections and quantifies its effect on their shape and area. Using the focus–directrix definition and standard Cartesian and polar equations, we derive relationships between eccentricity and conic parameters (a, b, c) and verify classification ranges: circle (e = 0), ellipse (0 < e < 1), parabola (e = 1), and hyperbola (e > 1). Analytic derivations and computational examples show that for ellipses $A = \pi a^2\sqrt{(1-e^2)}$, so increasing e from 0 toward 1 reduces area and increases elongation, while for hyperbolas larger e produces wider branch separation and steeper asymptotes. Comparative examples demonstrate that figures with the same e are similar in shape but differ in scale, confirming e controls shape while a and b set size. The study concludes that eccentricity is the principal numerical and structural identifier for classifying conic sections and links these geometric results to applications in orbital mechanics and engineering.

Keywords: Eccentricity, Classification, Elongation, Area, Applications

Introduction

Finding patterns, relationships, and structures in both the physical world and the abstract world of numbers is at the heart of mathematics. One of its major branches, geometry, allows us to study shapes, their dimensions, and how these dimensions influence measurable properties such as area. Among these shapes, conic sections have long captured the interest of mathematicians because of their fundamental role in describing natural phenomena and their mathematical elegance.

Conic sections circles, ellipses, parabolas, and hyperbolas are curves formed by the intersection of a plane with a double cone. This uniformity in their geometric definition makes them not only visually appealing but also analyzable through a unified mathematical framework. Common examples include planetary orbits (ellipses), satellite antennas (parabolas), and the paths of comets (highly elliptical orbits). One of the most important properties of a conic section is its eccentricity (e), which serves as the distinguishing parameter that classifies one type of conic from another. As eccentricity varies, the shape of the conic section changes dramatically; however, the exact relationship between eccentricity values and conic classification is not always immediately obvious to students.

To better understand this relationship, mathematicians use the focus-directrix definition of conic sections. The most fundamental definition expresses eccentricity as:

$$e = \frac{distance\ to\ focus}{distance\ to\ directrix} = \frac{PF}{PD}$$

Where PF the distance from any point on the conic to its focus, and PD is the distance from that same point to its directrix. In analytic geometry, this definition leads to the standard equations for each conic type through algebraic derivation. For instance, the equation of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ can be derived directly from the eccentricity relationship $e = \frac{c}{a}$ and $c^2 = a^2 - b^2$ (Briggs *et al*., 2022) [4] This formula shows that eccentricity is a constant ratio for each conic section, highlighting the relationship between a conic's geometric properties and its classification.

The study of conic sections is deeply connected to calculus when analyzing how these curves behave under transformation and optimization. Calculus provides tools for examining rates of change and determining how small variations in eccentricity affect the shape of conic curves. For example, finding the derivative of the area of an ellipse with respect to its eccentricity reveals how efficiently the ellipse encloses space as it becomes more elongated. These concepts are fundamental when studying parametric equations, polar coordinates, and optimization problems involving conic sections (Briggs *et al*., 2022) [4].

Another important concept related to this topic is the classification system for conic sections based on eccentricity values. Studies show that among all possible values of ee, each conic section occupies a distinct range: a circle has $e = 0$; an ellipse has $0 < e < 1$; a parabola has $e = 1$; and a hyperbola has $e > 1$. As the eccentricity increases within the elliptical range, the ellipse becomes progressively more elongated, deviating further from circular shape. This concept is studied in topics such as orbital mechanics and celestial dynamics, showing how simple geometric parameters connect to complex physical phenomena (Physics Frontier, 2025) [9].

Supporting this idea, research indicates that eccentricity is vital for understanding orbital paths in celestial mechanics. Earth's nearly circular orbit has low eccentricity ($e \approx 0.0167$), while comets often exhibit highly elliptical orbits with much higher eccentricity values (Physics Frontier, 2025) [9]. Other studies also highlight the importance of eccentricity in engineering applications, emphasizing its role in the design of parabolic reflectors, telescopes, and elliptical lenses (Briggs *et al*., 2022) [4].

This investigation focuses on examining how eccentricity distinguishes and classifies different conic sections. By analyzing the mathematical definition of eccentricity, deriving standard equations for each conic type, and examining real-world applications in orbital mechanics, the study aims to identify patterns, develop generalizations, and gain a deeper understanding of the underlying geometry. Ultimately, this study seeks to connect theoretical concepts with practical applications, particularly in fields such as astronomy, engineering, and mathematical education.

Statement of the problem

This study aims to explore and describe the role of eccentricity in distinguishing conic sections.

Specifically, it seeks to answer the following questions:

- How does the area or shape of a conic section change as the eccentricity value increases?
- What pattern or relationship exists between the eccentricity value and the type of conic section?
- How do the different conic sections compare when they have the same eccentricity value?

Methods and Research design

Research design

This study employed a descriptive-analytical research design to investigate the structural patterns and algebraic properties of eccentricity in distinguishing conic sections (circle, ellipse, parabola, and hyperbola). The research focused on identifying recurring numerical and geometric patterns, analyzing their correspondence with the focus–directrix definition and standard conic equations, and exploring their applications in problem-solving and algebraic reasoning. A purely mathematical and analytical approach was used, relying on symbolic derivation, geometric construction, numerical computation, and visualization.

The descriptive-analytical design was selected because it allows systematic description of eccentricity's classification role and analytical examination of algebraic–geometric relationships without human subjects or experimental manipulation.

Research materials

The materials used in this study included:

1. **Conic definitions and equations**

- Standard forms of conic section equations in Cartesian and polar coordinates
- Focus–directrix property definitions: eccentricity

$$e = \frac{\text{distance to focus}}{\text{distance to directrix}}$$

- Parametric representations of circles, ellipses, parabolas, and hyperbolas

2. **Mathematical texts and references**

- Scholarly textbooks on analytic geometry and conic sections (e.g., Briggs *et al*., 2022; Dobbs, 2024) [4, 5]
- Historical texts and mathematical literature on conic classification and orbital mechanics
- Peer-reviewed journal articles published after 2021 on eccentricity and conic sections (Siegel, 2021; Kallergis, 2024; Dintarini *et al*., 2024) [10, 8, 6].

3. **Software tools**

- GeoGebra- Interactive geometric construction and focus–directrix animation

Research methods

The study followed these steps:

1. **Construction and definition of conic sections**

- Conic sections (circle, ellipse, parabola, hyperbola) were defined using the focus–directrix property, where eccentricity e is expressed as the ratio of the distance from any point on the conic to its focus divided by the distance from that point to its directrix.
- Standard Cartesian and polar equations for each conic type were established to serve as the mathematical framework for analysis.

2. **Pattern identification and classification analysis**

Classification patterns based on eccentricity values were identified: circle ($e = 0$), ellipse ($0 < e < 1$), parabola ($e = 1$), and hyperbola ($e > 1$) through manual derivation and computational verification.

- Geometric patterns, including ellipse flattening, parabola symmetry, and hyperbola asymptotes, were examined to determine how they change as eccentricity varies across different ranges.

3. **Algebraic property examination**

- The correspondence between eccentricity and conic parameters (semi-major axis a, semi-minor axis b, focal distance c) was analyzed using algebraic derivations: $e = c/a$, $c^2 = a^2 - b^2$ (ellipse), and $c^2 = a^2 + b^2$ (hyperbola).
- The polar equation $r = \frac{ed}{1 \pm e\cos\theta}$ was examined to verify how varying e affects curve shape and asymptotic behavior, confirming predictable algebraic relationships.

4. **Application and Interpretation**

- Identified patterns and eccentricity–conic relationships were interpreted in terms of their mathematical applications, including orbital mechanics (Earth's low eccentricity $e \approx 0.0167$, comets' high eccentricity), engineering design (parabolic reflectors, telescopes, elliptical lenses), and conic classification from quadratic equations.

- The practical relevance of understanding eccentricity was discussed, highlighting its usefulness in simplifying conic identification, enhancing geometric reasoning, and connecting theoretical concepts to real-world applications in astronomy, engineering, and mathematical education.

Data analysis

The data analysis of this study focused on identifying and describing the structural patterns and algebraic properties of eccentricity in distinguishing conic sections. The analysis was based on the focus-directrix definition, the standard equations of conic sections, and the relationship between eccentricity and geometric form. The findings were organized according to the Statement of the Problem and interpreted through mathematical comparison, classification, and visualization.

Table 1: Classification of conic sections by eccentricity

Conic type	Eccentricity (e)	Range description	Shape characteristic
Circle	e = 0	Single value	Perfectly round
Ellipse	0 < e < 1	0 to 1 (exclusive)	Elongated circle
Parabola	e = 1	Single value	Open single-branch curve
Hyperbola	e > 1	Greater than 1	Two-branch open curve

This table shows that each conic section has a distinct range of eccentricities. The classification pattern confirms that eccentricity is the main property used to distinguish one conic type from another.

Pattern in ellipses

For ellipses, the data show that as eccentricity increases, the curve becomes more elongated and the area decreases. The semi-minor axis also becomes smaller, which causes the ellipse to flatten. At e=0, the shape is a circle, while values closer to e=1 produce very narrow ellipse.

Eccentricity e	Relative shape change
0.00	Circle with maximum area
0.30	Slightly elongated ellipse
0.60	Moderately elongated ellipse
0.80	Highly elongated ellipse
0.90	Very narrow ellipse
0.99	Nearly flat ellipse

This pattern shows that ellipse shape is strongly influenced by eccentricity, and the change is gradual and predictable.

Pattern in hyperbolas

For hyperbolas, the analysis shows that as eccentricity increases beyond 1, the branches spread farther apart and the curve opens more widely. The asymptotes become steeper as eccentricity grows. This indicates that eccentricity also controls the degree of openness of a hyperbola.

Eccentricity e	Relative shape change
1.00	Parabola boundary
1.20	Slightly open hyperbola
1.50	Moderately open hyperbola
2.00	Wider hyperbola
3.00	Very open hyperbola

This result confirms that hyperbola shape changes systematically as eccentricity increases.

Comparison of same eccentricity values

The data also show that different conic types do not share the same eccentricity value except at boundary cases. A circle is the only conic with $e = 0$, and a parabola is the only conic with $e = 1$. Ellipses and hyperbolas occupy separate eccentricity intervals, so their values do not overlap.

Eccentricity value	Possible conic type	Observation
$e = 0$	Circle only	Unique boundary case
$0 < e < 1$	Ellipse only	Shapes are similar within the same type
$e = 1$	Parabola only	Unique boundary case
$e > 1$	Hyperbola only	Shapes are similar within the same type

This comparison confirms that eccentricity can be used to classify conic sections accurately and consistently.

Results and Discussion

This chapter presents the results of the analytical investigation on the role of eccentricity in distinguishing conic sections. The discussion is organized according to the Statement of the Problem and the methods employed in the study, focusing on how eccentricity affects the shape of conic sections, how it classifies circles, ellipses, parabolas, and hyperbolas, and how it supports geometric and algebraic interpretation. The findings demonstrate that eccentricity is a fundamental parameter in identifying and describing conic sections. Each section provides detailed analysis supported by mathematical principles, showing how one value can explain the geometric behavior of different conic forms.

1. Changes in shape as eccentricity increases

The investigation shows that the shape of a conic section changes in a predictable way as eccentricity increases. For an ellipse, increasing eccentricity makes the curve longer and narrower because the semi-minor axis becomes smaller. Consequently, the enclosed area decreases.

The area of an ellipse is $A = \pi ab$, where $b = a\sqrt{(1-e^2)}$. Thus, $A = \pi a^2\sqrt{(1-e^2)}$.

Example:

Let a = 10.

For e = 0.60:

$b = 10\sqrt{(1-0.60^2)}=8$

Area = $\pi(10)(8)=80\pi\approx 251.33$

For e = 0.80:
$b = 10\sqrt{(1-0.80^2)}=6$
Area = $\pi(10)(6)=60\pi\approx 188.50$

For ellipses, the curve becomes more elongated as $e$increases from 0 to 1. The area decreases, the semi-minor axis becomes smaller, and the ellipse becomes flatter. At $e = 0$, the ellipse is a circle with maximum area, while values close to 1 produce a very narrow curve.

For hyperbolas, c = ae.
Let a = 5.
If e = 1.20, c = 6.
If e = 2.00, c = 10.

For hyperbolas, the opposite trend is observed. As eccentricity increases beyond 1, the branches spread farther apart and the asymptotes become steeper. This means that larger eccentricity values produce a more open curve. These patterns show that eccentricity directly controls the geometric form of the conic section.

2. Relationship between eccentricity and conic type

A clear relationship was identified between eccentricity values and conic type. The results confirm the classification pattern: a circle has $e = 0$, an ellipse has $0 < e < 1$, a parabola has $e = 1$, and a hyperbola has $e > 1$. This pattern proves that each conic section occupies its own eccentricity range.

The investigation confirms that each conic section corresponds to a unique eccentricity range:
Circle: $e = 0$
Ellipse: $0 < e < 1$
Parabola: $e = 1$
Hyperbola: $e > 1$

Example 1
Given a = 10 and c = 5,
e = c/a = 5/10 = 0.5
Since 0 < 0.5 < 1, the conic is an ellipse.

Example 2
Given a = 8 and c = 8,
e = 8/8 = 1
Since e = 1, the conic is a parabola.

Example 3
Given a = 6 and c = 15,
e = 15/6 = 2.5
Since 2.5 > 1, the conic is a hyperbola.

This finding demonstrates that eccentricity is the key basis for distinguishing one conic section from another. The value of $e$does not merely describe a curve; it determines what type of conic it is. Therefore, eccentricity functions as both a numerical and structural identifier.

3. Comparison of conic sections with the same eccentricity

The investigation also compared conic sections having the same eccentricity. The results show that eccentricity determines the shape of a conic, while its dimensions determine its size.

Example
Ellipse A:
a = 10
e = 0.60
c = ae = (10)(0.60)=6
$b = \sqrt{(10^2-6^2)}$
$= \sqrt{64}$
= 8

Ellipse B:
a = 20
e = 0.60
c = ae = (20)(0.60)=12

$b = \sqrt{(20^2-12^2)}$
$= \sqrt{256}$
=16

Comparison
Ellipse A: a=10, b=8, e=0.60
Ellipse B: a=20, b=16, e=0.60

Although Ellipse B is twice as large, both ellipses have the same eccentricity. Therefore, they have the same degree of elongation and differ only in size.
The results show that different conic types cannot have the same eccentricity value except in boundary cases. A circle is the only conic with $e = 0$, and a parabola is the only conic with $e = 1$. Ellipses and hyperbolas also do not overlap because their eccentricity ranges are separate. This confirms that eccentricity uniquely distinguishes the four conic sections.
Within the same conic type, figures with the same eccentricity are similar in shape even if their sizes differ. For example, ellipses with the same eccentricity have the same degree of flattening, although their actual dimensions may vary. This means that eccentricity determines shape similarity, while scale determines size.

Discussion

The findings of the study show that eccentricity plays an important role in determining both the shape and classification of conic sections.
For the first statement of the problem, which examined how the area or shape of a conic section changes as eccentricity increases, the results show that the changes follow a clear mathematical pattern. For ellipses, increasing the eccentricity from 0 toward 1 makes the curve more elongated while reducing its enclosed area. This occurs because the semi-minor axis becomes shorter as the eccentricity increases. The computed examples support this relationship, where higher eccentricity values produced smaller areas and narrower ellipses. In contrast, for hyperbolas, increasing the eccentricity

beyond 1 causes the branches to spread farther apart, resulting in a wider opening. These findings indicate that eccentricity directly influences the geometric shape of conic sections in a predictable manner.

For the second statement of the problem, which identified the relationship between eccentricity and the type of conic section, the investigation confirmed that each conic section has a unique eccentricity range. A circle has an eccentricity of 0, an ellipse has an eccentricity between 0 and 1, a parabola has an eccentricity of exactly 1, and a hyperbola has an eccentricity greater than 1. Since these ranges do not overlap, eccentricity provides a clear and consistent basis for classifying conic sections. The computed examples further showed that determining the eccentricity value is sufficient to identify the corresponding conic section.

For the third statement of the problem, which compared conic sections with the same eccentricity value, the findings show that conic sections of different types cannot have the same eccentricity except at the boundary values of 0 and 1. Within the same conic type, however, figures with the same eccentricity have the same shape even when their dimensions are different. The comparison between two ellipses with an eccentricity of 0.60 demonstrated that both had the same degree of elongation despite having different sizes. This indicates that eccentricity determines the shape of a conic section, while its dimensions determine its overall size.

Conclusion

Based on the findings of the investigation, it is concluded that eccentricity is the key property that distinguishes conic sections. As the eccentricity increases, the shape of a conic section changes in a predictable way. For ellipses, increasing eccentricity produces a more elongated shape and decreases the enclosed area, while for hyperbolas, it results in a wider opening.

The investigation also confirms that each conic section corresponds to a unique eccentricity range. A circle has an eccentricity of 0, an ellipse has an eccentricity between 0 and 1, a parabola has an eccentricity of 1, and a hyperbola has an eccentricity greater than 1. These distinct ranges allow conic sections to be classified accurately using eccentricity alone.

Furthermore, conic sections with the same eccentricity within the same type have the same shape but may differ in size. This shows that eccentricity determines the geometric form of a conic section, while its dimensions determine its scale.

Overall, the study answers the research questions by showing that eccentricity explains the changes in shape, provides the basis for classifying conic sections, and identifies similarities among conics with the same eccentricity. These findings highlight the importance of eccentricity in understanding the mathematical properties of conic sections.

Recommendation

Based on the findings, the study recommends that teachers use eccentricity as a core concept when introducing conic sections in mathematics lessons. Visual tools such as graphing activities and dynamic geometry software should also be used to help students better understand how changes in eccentricity affect the shape of circles, ellipses, parabolas, and hyperbolas. Future researchers may expand this investigation by exploring more real-life applications of conic sections in astronomy, engineering, and physics. It is also recommended that further studies examine how students learn and compare conic sections when interactive and traditional teaching methods are used. This will help strengthen both conceptual understanding and practical appreciation of the topic.

References

1. Briggs WH, Cochran L, Knill N. Briggs Calculus: Early Transcendentals. 3rd ed. Pearson, 2022.
2. Harvard Math Fun Facts. Eccentricity of Conics. Harvard College Mathematics Research. Available from: https://math.hmc.edu/funfacts/eccentricity-of-conics/
3. Physics Frontier. What is the Eccentricity of a Conic Section? YouTube, 2025. Available from: https://www.youtube.com/watch?v=CUxSCX56h18
4. Briggs WH, Cochran K, Gillett P. How Does the Eccentricity Determine the Type of Conic Section? In: Briggs Calculus: Early Transcendentals. 3rd ed. Pearson, 2022. Available from: https://www.pearson.com/channels/calculus/textbook-solutions
5. Dobbs DE. Reflecting on parabolas. Moroccan J Algebra Geom Appl. 2024;3(1):45-77. Available from: https://ced.fst-usmba.ac.ma/p/mjaga/wp-content/uploads/2024/06/MJAGA_Vol3_Iss1_4_45_77_Bis.pdf
6. Dintarini M, Fuad Y, Budiarto MT. Examining undergraduate students' abstraction of conic sections in a dynamic geometry environment. J Math Educ. 2024;15(3):717-734. Available from: https://doi.org/10.22342/jme.v15i3.pp717-734
7. GeoGebra. GeoGebra Documentation: Geometry Tools. Version 5. GeoGebra, 2025. Available from: https://www.geogebra.org/help
8. Kallergis N. EllipsoHyperbola: A common approach that joins the conic sections in 2D and 3D space. Eur J Math Stat. 2024;5(3):9-23. Available from: https://doi.org/10.24018/ejmath.2024.5.3.290
9. Physics Frontier. Eccentricity in Orbital Mechanics and Celestial Dynamics. Physics Frontier, 2025. Available from: https://www.physicsfrontier.com
10. Siegel L. Crafting conic sections. Math Horizons. 2021;29(2):29. Available from: https://doi.org/10.1080/10724117.2021.1978760
11. Taggart M. Eccentricity. Wolfram MathWorld, 2024. Available from: https://mathworld.wolfram.com/Eccentricity.html
12. Trochim W. Research Design. SDSU Research Methodology Education Product, 2005. Available from: https://ori.hhs.gov/education/products/sdsu/res_design.htm
13. Wolfram Research Inc. Wolfram Mathematica Documentation. Version 14. Wolfram Research, 2025. Available from: https://reference.wolfram.com.